MATH 303 – Measure Theory
Homework 4

Please upload a pdf of your solutions by 23:59 on Monday, November 3. The assignment will be graded out of 10 points, taking into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. Let λ be the Lebesgue measure on $\mathbb{R}$ (i.e., the Lebesgue–Stieltjes measure with distribution function $F(x) = x$). Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue-measurable and integrable over $[a, b]$, and

$$\underbrace{\int_a^b f(x)\ dx}_{\text{Riemann integral}} = \underbrace{\int_{[a,b]} f\ d\lambda}_{\text{Lebesgue integral}}$$

Solution: The Riemann integral is defined in terms of step functions, so let us first check that step functions are Lebesgue-measurable and integrate appropriately. Step functions are the special case of simple functions built from intervals rather than general measurable sets. Hence, step functions are Borel-measurable. Given a step function $s = \sum_{j=1}^n c_j \mathbb{1}_{(x_{j-1}, x_j]}$ with $a = x_0 < x_1 < \ldots < x_n = b$, we have

$$\int_{[a,b]} s\ d\lambda = \sum_{j=1}^n c_j \lambda\left((x_{j-1}, x_j]\right) = \sum_{j=1}^n c_j (x_j - x_{j-1}) = \int_a^b s(x)\ dx.$$

Suppose now that $f : [a, b] \to \mathbb{R}$ is an arbitrary Riemann integrable function. Then there exist sequences of step functions $\ell_n : [a, b] \to \mathbb{R}$ and $u_n : [a, b] \to \mathbb{R}$ such that $\ell_n \le f \le u_n$ and

$$\lim_{n\to\infty} \int_a^b \ell_n(x)\ dx = \lim_{n\to\infty} \int_a^b u_n(x)\ dx = \int_a^b f(x)\ dx.$$

(The functions u_n are giving better and better approximations of the upper Riemann integral and ℓ_n are approximating the lower Riemann integral.) Defining $\ell'_n = \max_{j\le n} \ell_j$ and $u'_n = \min_{j\le n} u_j$, we have that:

- $\ell'_1 \le \ell'_2 \le \ldots \ell'_n \le \ldots \le f \le \ldots \le u'_n \le \ldots \le u'_2 \le u'_1$,
- ℓ'_n and u'_n are step functions (since the maximum or minimum of finitely many step functions is again a step function), and
- $\lim_{n\to\infty} \int_a^b \ell'_n(x)\ dx = \lim_{n\to\infty} \int_a^b u'_n(x)\ dx = \int_a^b f(x)\ dx$ by monotonicity of the Riemann integral and the squeeze theorem.

Let $\ell(x) = \lim_{n\to\infty} \ell'_n(x) = \sup_{n\in\mathbb{N}} \ell_n$, and let $u(x) = \lim_{n\to\infty} u'_n(x) = \inf_{n\in\mathbb{N}} u_n(x)$. Since step functions are Borel measurable and limits of measurable functions are measurable, we have that ℓ and u are Borel-measurable functions. Moreover, by the dominated convergence theorem,

$$\int_{[a,b]} (u - \ell)\ d\lambda = \lim_{n\to\infty} \int_{[a,b]} (u_n' - \ell_n')\ d\lambda = \lim_{n\to\infty} \int_a^b (u_n'(x) - \ell_n'(x))\ dx = 0.$$

Thus, Proposition 3.23, we conclude that $u = \ell$ a.e. But $\ell \leq f \leq u$, so $f = u = \ell$ a.e. Since λ is a complete measure and u and ℓ are measurable functions, it follows that f is Lebesgue measurable (see Problem 2 from Exercise Sheet #4). Finally, by Proposition 3.22,

$$\int_{[a,b]} f\ d\lambda = \int_{[a,b]} u\ d\lambda = \lim_{n\to\infty} \int_{[a,b]} u_n'\ d\lambda = \lim_{n\to\infty} \int_a^b u_n'(x)\ dx = \int_a^b f(x)\ dx.$$